Sutra: Tensor-Op RNNs as a Compilation Target for Vector Symbolic Architectures

Emma Leonhart
emma@topazcomputing.com

Abstract

Sutra is a typed, purely functional programming language whose compiled forward pass is a PyTorch neural network. The compiler beta-reduces the whole program — primitives, control flow, string I/O — to a single substrate-pure tensor-op dataflow graph over a frozen embedding substrate (every operation is a tensor op; the language has no scalar-readout escape hatch). Rotation binding, unbind, bundle, polynomial Kleene three-valued logic, and tail-recursive loops all lower to tensor operations; the Kleene connectives are Lagrange-interpolated polynomials exact on the $\{-1, 0, +1\}$ truth grid.

Validation is one fact tested two ways. (1) The same program runs on four frozen embeddings spanning two modalities — three text encoders (nomic-embed-text, all-minilm, mxbai-embed-large) and one protein language model (ESM-2) — and decodes bundles at 100% accuracy through width k=8 on every substrate, where the textbook Hadamard product has already collapsed (2.5% on mxbai-embed-large, 7.5% on all-minilm). (2) PyTorch autograd flows through the actually compiled graph: a fuzzy-rule classifier written in `.su` trains from random init (18.7 ± 9.5%; chance = 20%, five classes) to 100.0 ± 0.0% (three seeds) by backpropagating through the emitted graph, the symbolic source unmodified. A weighted variant additionally trains a scalar cosine gain and writes it back into the `.su` source as a numeric literal; recompiling reproduces the trained behaviour to $\approx 2\times10^{-7}$ per logit, so the trained model is itself legible, recompilable code.

The same artifact is therefore both a logic program and a trainable neural network.

1 Introduction

A frozen embedding model maps strings (or amino-acid sequences, or any other input the model was trained on) into a deterministic continuous vector space. Given such a substrate, two technical questions follow:

1. **Which operations on these embeddings are reliable enough to be used as primitives** of a compositional algebra over the substrate's vector space?
2. **What is the correct binding operation?** Hyperdimensional computing's textbook bind operators (Hadamard product, circular convolution) were derived assuming hypervectors drawn from a controlled random distribution. Frozen LLM embeddings are not such a distribution: they are strongly *anisotropic*, concentrating in a narrow cone so cosine similarity is compressed and inflated even between unrelated items (Ethayarajh 2019). §3.2 measures four substrates and reports that rotation binding decodes at 100% accuracy through bundle widths where Hadamard has already collapsed.

Preprint.

This paper answers both questions in the form of a working programming language, **Sutra**, whose primitives are these consolidated operations and whose compiled forward pass is a PyTorch neural network. The naming: **Sutra** is the Sanskrit *sūtra* (thread, rule, aphorism), the term for Pāṇini's foundational Sanskrit grammar.

1.1 Contributions

The four core technical contributions of this paper are:

1. **Polynomial fuzzy logic via Lagrange interpolation of Kleene's three-valued truth tables.** The truth axis encodes $T = +1$, $U = 0$, $F = -1$. On the discrete $\{-1, 0, +1\}$ grid, the Kleene connectives are $\mathrm{AND} = \min$, $\mathrm{OR} = \max$, $\mathrm{NOT} = -\cdot$. The min/max forms (the standard Gödel t-norm/t-conorm choice; Hájek 1998) are non-differentiable at the diagonal $a = b$, which breaks gradient flow when connectives compose with the tensor-op graph (van Krieken, Acar & van Harmelen 2022 survey the issue across t-norm-derived neural-symbolic operators). Sutra resolves this by Lagrange-interpolating each connective as a polynomial that is exact on the 3×3 Kleene grid and C^∞ elsewhere:

$$\begin{aligned}
\mathrm{AND}(a,b) &= \tfrac{1}{2}(a + b + ab - a^2 - b^2 + a^2b^2) \\
\mathrm{NAND}(a,b) &= \tfrac{1}{2}(-a - b - ab + a^2 + b^2 - a^2b^2) \\
\mathrm{OR}(a,b) &= \tfrac{1}{2}(a + b - ab + a^2 + b^2 - a^2b^2) \\
\mathrm{NOR}(a,b) &= \tfrac{1}{2}(-a - b + ab - a^2 - b^2 + a^2b^2) \\
\mathrm{NOT}(a) &= -a \\
\mathrm{XOR}(a,b) &= -ab \\
\mathrm{XNOR}(a,b) &= ab
\end{aligned}$$

   {AND, OR, NOT} is functionally complete for the Kleene fragment; NAND and NOR are just $-$AND and $-$OR (negation is $-\cdot$), and XOR/XNOR collapse to a single multiplicative term because their interpolant is zero whenever either input is U and bilinear in the {−1, +1} corners. Every Kleene-valid connective is therefore a polynomial tensor-op-graph fragment, gradient-compatible, branchless, and exact on the discrete-logic regime. A symbolic if-then rule built from these gates is one fused subgraph that PyTorch autograd backprops through end-to-end (§3.6).

   The choice of Lagrange interpolation over softer alternatives — softened t-norms such as the Einstein or Yager families, or soft-min/soft-max with a temperature — is fixed by a non-negotiable requirement: agreement with classical Kleene reasoning on the discrete grid must be bit-for-bit, not approximate. Softened t-norms drift from $\{-1, 0, +1\}$'s truth values at the grid corners themselves; Lagrange interpolation on the 3×3 Kleene grid is the unique polynomial that *exactly* recovers each connective at the nine grid points while remaining C^∞ off it. The trade-off is one of monotonicity: the bilinear--quadratic interpolant is non-monotonic at some interior points (a defuzzified boolean that happens to read $a = 0.5,\ b = 0$ produces $\mathrm{AND}(a,b) = 0.125$, peaking off-grid before dropping back to 0 at $a = 1$). Sutra accepts this in exchange for grid-exactness; replacing the interpolant with a higher-degree monotonic polynomial is a future direction.

2. **Beta reduction to a substrate-pure tensor-op graph.** The compiler inlines stdlib operator definitions, beta-reduces through bound names, then runs an algebraic-simplification pass over the residual. What's left is a fused tensor-op graph (matmul / element-wise / nonlinear) with no named bindings or function calls. Three concrete moves go beyond standard inlining + constant folding: conditionals lower to soft-mux polynomials ($\frac{1+\mathrm{cond}}{2}\, a + \frac{1-\mathrm{cond}}{2}\, b$) so the compiled artifact has no `if` opcodes; Haar-orthogonal binding rotations `R_role` are materialized at compile time so runtime `bind` is one matmul against a constant matrix; canonical synthetic axes are assigned compile-time so every primitive-type read/write is a known index, not a hashtable lookup. §4.2 traces this lowering stage-by-stage on a concrete program; the compilation pipeline as a whole is diagrammed in Figure~3.

3. **Tail recursion as the loop primitive.** Loops are tail-recursive function declarations (`do_while`, `while_loop`, `iterative_loop`, `foreach_loop`) whose body's `return`

`NAME(args)` becomes the recurrent step. Each loop compiles to a soft-halt RNN cell with substrate-pure halt detection (heaviside → cumulative monotone halt → soft-mux state freeze). The body of every loop tick is one straight-line tensor pipeline with no in-graph branches; a thin Python `while True: ... break` driver wraps the body and terminates when the halt scalar saturates (§3.4). The state vector is fixed-width across iterations, **O(1) state, O(N) compute, O(N) gradient tape during training**, where N is iterations actually executed.

4. **Synthetic-dimension rotation binding as an angular hash map.** The compiler reserves a synthetic block of canonical dimensions and uses Haar-orthogonal rotations seeded from the role's content hash to bind keys to slots. We are unaware of any prior use of a high-dimensional rotation pattern as the substrate for a functional hash-map primitive.

These four primitives integrate into a single working compiler that lowers `.su` source to a self-contained PyTorch module on CPU or CUDA. Program inputs and outputs are embeddings in the substrate's vector space; a compile-time codebook (implemented with an embedded vector database, §3.5) handles the convenience of source-level string literals and nearest-string output.

1.2 The substrate is the architecture target

A Sutra program is compiled for an *embedding-space architecture*, the way a C program is compiled for x86 and a CUDA kernel for an NVIDIA SM. The embedding model fixes dimensionality, the geometry of the semantic block, and the meaning of every basis-vector lookup; swap the model and the same source recompiles to a different `.sdb` codebook against a different geometry. The substrate need not be an LLM, it can be any network producing a dense vector representation, including the hidden state of a trained model. §3.2's ESM-2 protein-LM row demonstrates this substrate-agnostically.

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2 Related Work

2.1 Vector Symbolic Architectures

VSA is a family of algebraic frameworks for computing with high- dimensional vectors (Kanerva 2009; Plate 1995; Gayler 2003). The standard VSA development assumes hypervectors drawn from a controlled random distribution designed for the algebra; bind is typically Hadamard product or circular convolution. Frozen LLM embedding spaces are not designed for VSA: they are anisotropic (Ethayarajh 2019; Gao et al. 2019; Mu et al. 2018) — representations concentrate in a narrow cone, so cosine similarity has low dynamic range and the textbook bind operations do not transfer cleanly. The same anisotropy makes an unweighted cosine read a weak rule signal, which is why §3.6 trains the embeddings the rules compare against rather than relying on raw cosine. Rotation binding (`R_role @ filler` for a role-seeded Haar-random orthogonal `R_role`) is the choice that worked across the substrates we tested, and is what Sutra uses today; §3.2 reports the per-substrate measurements supporting that choice.

Recent work gives VSAs a category-theoretic foundation: Shaw et al. (2025) formalize the binding, bundling, and similarity operations within a categorical framework and, discussing open directions, identify the construction of a *purely functional* VSA — functional in the programming-language sense — as a significant engineering goal. Sutra can be read as an independent, engineering-first realization of that direction: a functional language whose primitives are VSA operations, compiled to a differentiable neural-network forward pass. We arrived at the construction before encountering their formalization; that two routes — categorical axiomatization and engineering-first iteration — converge on the same object is some evidence the direction is structurally natural.

The closest software peer in the VSA space is **TorchHD** (Heddes et al. 2023), a PyTorch library that exposes VSA primitives (bind, bundle, similarity) as tensor operations. Sutra and TorchHD differ on what the user writes and what the compiler does:

- **TorchHD is a *library*.** The user writes Python code that calls TorchHD primitives; control flow is host-side Python; there is no source-language layer above the primitives, no compile

step, and no algebraic reduction across primitive calls. Each primitive call is a tensor op, but the program itself is a Python function with whatever control flow the user wrote.
- **Sutra is a *language with a compiler*.** The user writes `.su` source which the compiler beta-reduces to a substrate-pure tensor-op graph (§1.1-2): a single straight-line graph of matmul / element-wise / nonlinear ops with no Python control flow. Loops are tail-recursive function declarations that lower to soft-halt RNN cells; conditionals are differentiable fuzzy interpolations rather than Python `if`. Hash-map structure is implemented via synthetic-dimension rotation, not via a host-side dictionary.

A second axis where Sutra differs from existing HDC software is **string I/O**. TorchHD and similar libraries expose the algebra over user-supplied hypervectors; the user maintains a `dict[str, hypervector]` and an explicit codebook tensor by hand. Sutra's compile-time codebook (§3.5) closes that loop: every embedded string in `.su` source is embedded once at compile time via the configured frozen LLM, stored in the project's `.sdb` codebook, and decoded at the program output via `nearest_string`. The frozen-LLM embedding is load-bearing, random hypervectors yield a working VSA algebra with no I/O story.

The structural differences (Sutra contains no Python, the string-to-vector map and codebook are constructed by the compiler rather than by the user, and the whole program reduces to a single fused tensor-op graph) are differences in artifact shape, not library speed.

2.2 Comparison to other neuro-symbolic languages

The closest neuro-symbolic-language peers are **Scallop** (Li et al. 2023, Datalog with provenance-semiring differentiability), **DeepProbLog** (Manhaeve et al. 2018, ProbLog with neural predicates), **Logic Tensor Networks** (Badreddine et al. 2022, first-order logic compiled to t-norm losses), and **NeurASP** (Yang et al. 2020, Answer Set Programming with neural predicates). All share a two-stage perception-then-reasoning shape: a neural model extracts discrete symbols from raw input, and a symbolic program reasons over those symbols. Sutra's shape is different at this architectural level: the substrate is a continuous embedding space throughout, primitives operate on vectors end-to-end, and the whole program (including what would be the logic program in Scallop) compiles to a single fused tensor-op graph through beta reduction. There is no discrete symbolic stratum to extract into or reason over; differentiability is inherited from the tensor-op graph itself, not from a provenance annotation on a relational query. The two are good at different problem structures: Scallop and its peers when the problem is naturally relational and perception cleanly factors out; Sutra when computation is best expressed as algebra on vectors over a substrate the program reads strings into and decodes strings out of.

The closest HDC peer with compiler infrastructure is **HDCC** (Vergés et al. 2023), a description-file DSL targeting self-contained C for embedded classification, random/level hypervectors only, no general control flow, scoped to classification. **TorchHD** and OpenHD / HDTorch are libraries without a language-level loop primitive. To the authors' knowledge (literature reviewed through early 2026), no published HDC system combines (a) one fused tensor-op graph as compile target, (b) HDC primitives as the operations, (c) a frozen externally-trained vector embedding space as the substrate, and (d) tail-recursive loops compiled to soft-halt RNN cells with constant state-vector width in recursion depth. The combination is what distinguishes Sutra, not any one of those properties in isolation.

2.3 Fuzzy logic and neuro-fuzzy systems

Sutra's polynomial connectives sit downstream of fuzzy set theory (Zadeh 1965) and the neuro-fuzzy lineage that makes fuzzy inference trainable — adaptive-network fuzzy inference systems (Jang 1993) and the broader fuzzy-neural-network family (Buckley & Hayashi 1994). Those systems *learn* the fuzzy logic itself: membership functions and rule parameters are the trainable object. Sutra inverts this. Its connectives are fixed Lagrange interpolants of Kleene's three-valued tables (§1.1-1), exact on the discrete grid and never tuned; the only learnable parameters are the embeddings the frozen rule graph evaluates against (§3.6). Sutra is therefore closer in spirit to the differentiable t-norm analysis of van Krieken, Acar & van Harmelen (2022) than to membership-function learning, and differs from both by compiling the connectives into a single substrate-pure tensor-op graph rather than evaluating them in a host interpreter.

2.4 Differentiable Programming, AOT Compilation, and Knowledge

Compilation

The closest design ancestors are partial-evaluation systems that specialize programs at compile time (the Futamura projections; Futamura 1971), differentiable programming systems that treat programs as differentiable functions (JAX), AOT compilation of neural networks (TVM, XLA), and knowledge compilation in symbolic AI (Darwiche & Marquis 2002). Sutra differs from each: TVM/XLA start from a network, not toward one; JAX treats programs as differentiable but does not bake source literals into weights; partial evaluation specializes for compile-time-known values but does not target a neural-network-shaped artifact; knowledge compilation targets Boolean circuits, not continuous embedding spaces. Sutra's combination (fold source literals into the weight structure, compile control flow to RNN cells, run the whole program as one tensor-op graph over a *continuous* substrate) is the novel position.

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3 Preliminaries

This section consolidates the two prerequisites a reader needs before §3: the vector-symbolic primitives Sutra compiles to, and the three-valued logic its conditionals and connectives are built on. Both are used heavily from §3 onward. The anisotropy of frozen embedding spaces — the other recurring fact — is introduced inline where it bears (§2.1, §3.2, §3.7) rather than here.

3.1 Vector-symbolic primitives

Sutra's data-structure primitives come from Vector Symbolic Architectures (hyperdimensional computing; Plate 1995; Gayler 2003; Kanerva 2009). Four operations carry the algebra. *Binding* composes a role with a filler into a single vector dissimilar to both; *unbinding* is its inverse, recovering the filler given the role. *Bundling* superposes several vectors into one that stays similar to each. *Similarity* (cosine) measures how close two vectors are, and *cleanup* uses it to decode a noisy or superposed vector back to the nearest stored item. Together these let a record, a sequence, or a graph be carried as one fixed-width vector and queried by vector arithmetic — the property Sutra's compiler builds data structures on. §3.1 and Appendix A give the exact operators Sutra uses: rotation binding (`R_role @ filler` for a role-seeded Haar-orthogonal `R_role`), additive bundling, and cosine similarity.

3.2 Kleene three-valued logic

Sutra's conditionals and boolean connectives are built on Kleene's *strong* three-valued logic (Kleene 1952), which extends classical {true, false} with a third value *unknown* for under-determined propositions. With the truth values ordered from false up through unknown to true, conjunction is min, disjunction is max, and negation reverses the order — the Gödel t-norm and t-conorm, the standard choice when the third value means "undetermined" rather than "half true". This reproduces classical logic whenever both inputs are determined, and propagates *unknown* exactly when the classical result depends on the unknown input: *true or unknown* is true, but *false or unknown* is unknown. Sutra encodes the three values antipodally on a dedicated truth axis — T = +1, U = 0, F = −1 — so min, max, and negation become numeric operations on that axis. The contribution in §1.1 is to replace the non-differentiable min/max with Lagrange-interpolated polynomials that are *exact* on the nine-point $\{-1, 0, +1\}$ truth grid and smooth off it; §3.6 then backpropagates through a symbolic rule built from these gates.

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4 Consolidation into Canonical Primitives

The central design move: hold the operation interface fixed and pick a binding implementation that works on dense externally-trained substrates. Standard VSA's Hadamard product fails here,

elementwise multiplication of correlated real-valued vectors produces destructive crosstalk on bundled retrieval (§3.2 measures this directly). Rotation binding works: each role gets a Haar-random orthogonal `R_role` seeded by `hash(role)`, and `bind(role, filler) = R_role @ filler` is invertible (unbind is the transpose) and well-conditioned. The compiler caches `R_role` per-role at module init so runtime bind is a single matmul against a precomputed matrix.

Each subsection below is tagged *method* (the consolidated primitives, extended-state-vector layout, loop cell, codebook, and type surface) or *experiment* (the cross-substrate capacity measurements, §3.2 / §3.2.1, and the end-to-end differentiable-training result, §3.6). The two are interleaved by topic but labelled so the evaluation is separable from the design.

4.1 Notation — method

We work in $\mathbb{R}^d$ with d the substrate's embedding dimension (768 for nomic-embed-text). Every value has the layout $[\,\text{semantic} \mid \text{synthetic}\,]$. The seven primitive operations: $\text{bind}(r, f) = R_r f$ where $R_r = \text{QR}(\text{hash}(r)).Q$ is Haar-orthogonal, $\text{unbind}(r, v) = R_r^\top v$, $\text{bundle}(x, y) = (x + y)/(\|x + y\| + \varepsilon)$, $\text{similarity}(x, y) = (x \cdot y)/(\|x\|\,\|y\| + \varepsilon)$, $\text{normalize}(v) = v/(\|v\| + \varepsilon)$, the Lagrange Kleene gates as in §1.1-1, and the soft-halt cell of §3.4. Full signature/definition table and the soft-halt cell update equations are in Appendix A.

4.2 Capacity of rotation versus Hadamard binding across substrates — experiment

We measure decode accuracy as a function of bundle width k on real embeddings across four substrates spanning two modalities: three frozen LLM text encoders (nomic-embed-text, all-minilm, mxbai-embed-large) and one frozen protein language model (ESM-2 small, `facebook/esm2_t6_8M_UR50D`). LLM substrates embed an 84-word noun vocabulary; the ESM-2 substrate embeds an 84-sequence amino-acid vocabulary (full protocol in Appendix C). For each bundle width and binding scheme we run 10 trials, sampling k random (role, filler) pairs without replacement, forming the bundle, and decoding by unbind + argmax-cosine against the full codebook. *Rotation binding* uses a role-seeded Haar-orthogonal `R_role`; *Hadamard binding* is the textbook elementwise product (MAP-VSA).

Cross-substrate decode accuracy at representative widths (full k ∈ {2, 4, 8, 16, 24, 32, 48} sweeps in Appendix C):

substrate (dim)	rotation k=8	rotation k=48	Hadamard k=8	Hadamard k=48
nomic-embed-text (768)	100.0%	93.3%	87.5%	48.3%
all-minilm (384)	100.0%	42.3%	7.5%	1.7%
mxbai-embed-large (1024)	100.0%	72.1%	2.5%	1.0%
ESM-2 (320)	100.0%	44.2%	28.7%	4.2%

ESM-2 (Lin et al., Science 2023) is a protein language model trained on UniRef with no natural-language exposure; the same rotation-vs-Hadamard pattern reproduces in that modality. Rotation reversibility round-trip across all four substrates: mean ‖unbind(R, bind(R, x)) − x‖ = 1.5×10^{-15} (floating-point round-off, Q orthogonal). Reproduction: `experiments/rotation_binding_capacity_{llm,bioinformatics}.py`.

4.2.1 Noise accumulation across chained bind/unbind cycles — experiment

The §3.2 protocol measures one bind+bundle+unbind cycle. Nested records (a recovered filler becoming the role of a sub-record) add bundle noise per level. We measured this directly: chain lengths L ∈ {1, 2, 4, 8, ...}, 20 trials, bundle width 4. Raw accuracy holds at 100% through L=2 on every substrate and falls to chance (1/84) by L=8. The demonstrated regime is therefore single-cycle records, which matches the shape of the `role_filler_record`, `knowledge_graph`, and predicate-lookup demos. Pure rotation chains without per-step distractor bundling remain exact (round-trip 1.5×10^{-15} per cycle), so the noise mechanism here does not apply to the soft-halt loop cell of §3.4. Reproduction script: `experiments/crosstalk_chain.py`; full per-substrate L-sweep tables in Appendix D.

4.3 The extended-state-vector layout — method

Every value carries a fixed `[semantic | synthetic]` layout: the d-dimensional semantic block holds the substrate embedding for vector-shaped values, and a small synthetic block reserves canonical axes for primitive types (real, imag, truth, char), a loop-completion flag, and promise/axon status flags. Variable slots hold **full extended-layout vectors** in a per-slot store keyed by compile-time slot indices — a scalar is carried as its number-vector (value on the real axis) and String/vector state is carried whole, so every slot type shares one representation. Rotation binding is block-diagonal across the split (`Q_role` is Haar-random in the semantic block, identity on the synthetic block), so the synthetic axes pass through bind/unbind unchanged, a fuzzy-truth scalar can coexist with a semantic vector inside the same value without bind smearing them. Full per-axis purpose table and slot allocator details in Appendix B.

4.4 First-class loops as RNN cells — method

Runtime data-dependent loops compile to **self-halting RNN cells**. Each tick: snapshot pre-step state, evaluate halt on the substrate (truth-axis read $\rightarrow$ heaviside $\rightarrow$ cumulative saturating sum to `halted`), run the cell body, soft-mux between pre- and new-step state by `halted`. A Python `while True:` driver breaks the moment `halted` saturates; this is the only host-side branch in the loop machinery. The halt read is a hard step (heaviside), which has no usable gradient: loop *termination* is a forward-only decision, and the differentiable-training results in this paper (§3.6) are on loop-free graphs. Inside the cell body, every operation is a substrate tensor op. No compile-time iteration cap, programs terminate when their halt condition fires. Standard PyTorch tracing handles a Python while-loop wrapping pure tensor ops; autograd records each iteration as it executes, which is the mechanism §3.6 relies on for backprop through the cell. Figure~1 visualizes one tick.

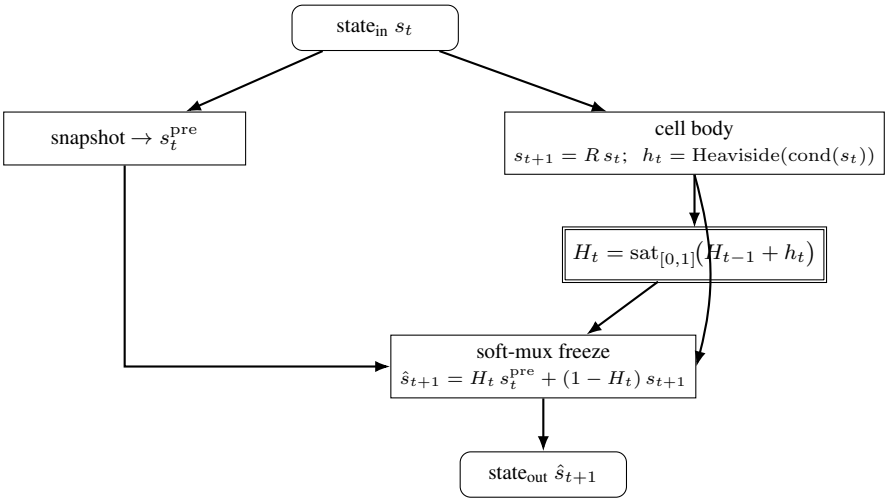

Figure 1: Per-tick dataflow of the soft-halt RNN cell. Once H_t saturates at 1, the soft-mux output equals s_t^{pre}, the loop has frozen. The cumulative halt H_t acts as a boundary read of the same shape as the codebook decode (§3.5).

Constant memory in recursion depth. The state vector is fixed-width and shared across iterations, so a tail-recursive loop consumes O(1) memory in the state vector regardless of trip count. Compute is O(N) and the autograd tape during training is O(N) in iterations actually executed (standard PyTorch, freed after backward). To the authors' knowledge no other HDC system or compiler exposes user-program-level recursion: HDCC is scoped to classification pipelines, TorchHD requires the user to write Python loops over hypervectors. The recurrent shape that emerges is the rational-weight RNN form whose computational power Siegelmann & Sontag (1992) characterized.

4.5 I/O is in the embedding space; the codebook is a comfort layer — method

A Sutra program's inputs and outputs are embeddings in the substrate's vector space. Strings are a convenience for writing source-level literals: every string literal in `.su` source is embedded once

at compile time and stored in a **codebook** (implemented as an embedded vector database with an HNSW index, on disk as a `.sdb` file shipped alongside the compiled module). At the program's output boundary, the runtime decode `_VSA.nearest_string(query)` maps a query embedding to the nearest stored string when the program's caller wants a string back. Calling the codebook at this boundary is shape-equivalent to calling PyTorch for a matmul, neither is the kind of host-side control flow substrate purity forbids. Implementation details (RDF triple layout, HNSW parameters, `.sdb` file format, complexity analysis) are in Appendix E.

4.6 End-to-end differentiable training through Sutra operations — experiment

Because every Sutra primitive compiles to a differentiable tensor operation, the compiled graph supports standard PyTorch `loss.backward()` without modification. We verify this by training learnable parameters through a fuzzy-logic classifier built entirely from Sutra operations.

Setup. The classifier is written in `.su` and compiled by the PyTorch codegen; the emitted module's `rule` function *is* the compiler output — `_VSA.similarity` composed with the Lagrange–Kleene AND/NOT polynomials, with no hand-written reimplementation. Five semantic classes, ten words each (50 inputs), embedded via nomic-embed-text (768-d, frozen). Five learnable prototype vectors are initialized randomly. The program compiled at $K = 5$ (generated by `gen_rule_su`; line-wrapped here, `.su` is whitespace-insensitive) is:

```
function fuzzy rule(vector x, vector own,
                    vector o0, vector o1, vector o2, vector o3) {
    return similarity(x, own)
        && !similarity(x, o0) && !similarity(x, o1)
        && !similarity(x, o2) && !similarity(x, o3);
}
```

so each class score is the compiled fuzzy rule

$$\text{rule}_i \;=\; \text{AND}\Big(\text{sim}(x, p_i),\; \bigwedge_{j \neq i} \text{NOT}\big(\text{sim}(x, p_j)\big)\Big)$$

with the AND-of-NOTs left-folded across the $K - 1$ other classes (at $K = 5$, four NOT terms folded under one AND). Full-batch cross-entropy over the five compiled rule scores drives Adam (Kingma & Ba 2015; defaults $\beta_1 = 0.9$, $\beta_2 = 0.999$, $\varepsilon = 10^{-8}$, learning rate 0.01) on the prototype embeddings, backpropagating through the emitted graph. 30 epochs, three seeds (0–2).

Results (3 seeds). From random-init accuracy at chance (18.7 ± 9.5%; chance = 20%), training reaches 100.0 ± 0.0% (mean ± s.d. over seeds 0–2; every seed reaches 100%); cross-entropy loss falls to ≈ 0.43. Every prototype receives a nonzero gradient, verified to propagate through the *emitted* graph (`_VSA.similarity` → the emitted Lagrange–Kleene NOT/AND → cross-entropy), not through a reimplementation.

Phase	Accuracy (mean ± s.d., n=3)
Before (random)	18.7 ± 9.5 %
After (30 ep)	100.0 ± 0.0 %

This experiment isolates gradient flow through the *compiled* symbolic graph: it trains and evaluates on the same 30-word set and reports in-sample accuracy purely as verification that backprop reaches every learnable prototype through the compiler's emitted ops — not a generalization claim; no held-out split. Only before/after accuracy was logged for the compiled run; per-epoch data was not recorded, so no intermediate-epoch curve is plotted.

Figure~2 shows the rule pipeline at $K = 3$, the smallest instance; the experiment uses the same shape at $K = 5$ — five cosine-similarity nodes against five learnable prototypes, the AND-of-NOTs folded over the four others. Each `sim_i` enters one branch of the AND-tree (rule i takes `sim_i` directly and $\text{NOT}(\text{sim}_j)$ for $j \neq i$); the rule scores are stacked, temperature-scaled, softmaxed,

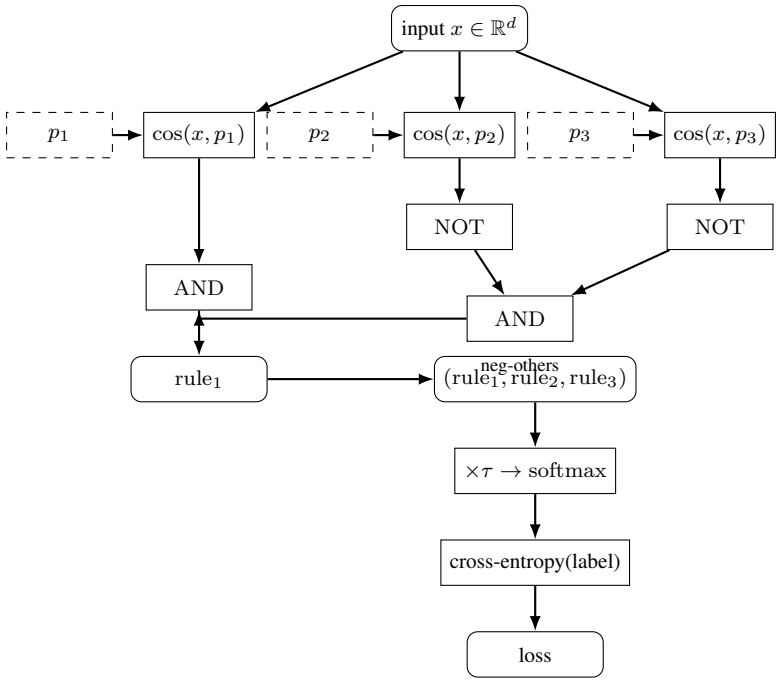

Figure 2: The $K = 3$ rule pipeline. Solid boxes are PyTorch tensor ops; dashed boxes are learnable prototypes. The AND in the leftmost branch combines $\cos(x, p_1)$ with the AND-of-NOTs over the other classes; rule_2 and rule_3 (omitted for clarity) have the symmetric shape. Every edge is a tensor; backprop reaches each p_i through this graph.

and cross-entropied. Every node is a compiler-emitted tensor op — no Python branches, no string-keyed lookup — so backprop reaches every prototype through *the same compiled graph that runs at inference* (literally so: the graph is the codegen output, not a reimplementation).

The AND-of-NOTs chain is a tensor pipeline that could naively saturate or vanish gradients; empirically it does not — every prototype receives a nonzero gradient and the five classes separate perfectly within 30 epochs, the symbolic program text unchanged across training. Standard `torch.autograd` suffices (no Sutra-specific autograd machinery) because the compiler emits only operations PyTorch already differentiates. The `.su` compiles once; the emitted `rule` is then evaluated over the batch with `torch.vmap` — a transform that runs the *same* compiled ops with a batch axis, not a reimplementation. Before training, the harness asserts the batched and per-sample evaluations agree within 10^{-4} on identical inputs and parameters, so the speedup is provably the identical compiled computation. Under this path the 5-class / 50-word / 30-epoch / 3-seed run takes $\approx$ 230 s on CPU and yields the numbers above (18.7 ± 9.5% $\rightarrow$ 100.0 ± 0.0%, three seeds). The earlier per-sample driver — one emitted `rule` call per class per input in a Python loop — produced the bit-identical result but took $\approx$ 6.2 h; that cost was Python-level call and per-sample autograd-graph overhead, not the compiled tensor math, and does not bound the achievable scale. Reproduction: `experiments/differentiable_training_compiled.py --batched` (compiles the `.su` once, vmaps the emitted `rule`, backprops the prototypes through the emitted graph; the equivalence assertion is on by default).

4.7 Trained weights compile back to legible source — experiment

Frozen LLM embedding spaces are *anisotropic*: learned token representations occupy a narrow cone rather than filling the sphere (the representation-degeneration effect; Gao et al. 2019, Ethayarajh 2019), so raw cosine similarities are squeezed into a compressed positive band — even unrelated pairs score well above zero. The Kleene AND/NOT decision in §3.6 depends on the *gap* between the matching cosine and the off-class cosines; when anisotropy compresses that band, the polynomial connectives have little dynamic range to separate classes. This motivates a weighted similarity —

`Equals(A, B, w)` — in which a single learned scalar gain w rescales the cosine term *before* the Kleene polynomials act, directly counteracting the anisotropic compression:

$$\text{rule}_i \;=\; \text{AND}\Big(w \cdot \text{sim}(x, p_i),\; \bigwedge_{j \neq i} \text{NOT}\big(w \cdot \text{sim}(x, p_j)\big)\Big)$$

Setup. The rule is written in `.su` with w declared as a `number` parameter and compiled by the PyTorch codegen; the emitted code uses w as a plain scalar multiplying the compiler-emitted `similarity`, so w both (i) trains through the emitted graph and (ii) is a single scalar that can be written back into source as a literal. Three semantic classes, eight words each (24 inputs), nomic-embed-text (768-d, frozen). The scalar gain w (initialized 1.0) **and** the three prototype vectors are trained through the emitted graph; full-batch cross-entropy, Adam (lr 0.02), 30 epochs, two seeds (0–1).

Results (2 seeds). From random-init accuracy at chance (33.3 ± 5.9%; chance = 33.3%), training reaches 100.0 ± 0.0% (every seed). The learned gain converges to $w^* = 1.434 \pm 0.004$ — both seeds move it the same way, from 1.0 to ≈ 1.43, sharpening the compressed cosine band rather than leaving it at the identity (a learned rescaling, not a tautology).

Phase	Accuracy (n=2)	Learned gain w^*
Before (random)	33.3 ± 5.9 %	1.000 (init)
After (30 ep)	100.0 ± 0.0 %	1.434 ± 0.004

Ablation: which trained parameter separates the classes? Running the *same* compiled graph with each factor frozen isolates its contribution (2 seeds, identical protocol):

condition	After accuracy (n=2)	Learned w^*
full (gain + prototypes)	100.0 ± 0.0 %	1.434 ± 0.004
prototypes only (w=1 frozen)	100.0 ± 0.0 %	1.000 (fixed)
gain only (prototypes frozen)	33.3 ± 5.9 %	0.421 ± 0.012

The prototypes carry the class separation: with the gain frozen at the identity they still reach 100%. The scalar gain *alone* cannot separate the classes — a single global rescaling of every similarity, applied to fixed random prototypes, leaves accuracy at chance (the gain drifts to ≈ 0.42 but never moves the argmax above $1/k$, since the nonlinear Kleene connectives still rank the unmoved prototypes the same way). So the jointly-trained $w^* \approx 1.43$ is a *co-adapted* parameter — consistent across seeds (hence not noise) but not the load-bearing factor for accuracy on this task. What this experiment establishes is the legibility round-trip below, not that the gain is necessary. Reproduction: `experiments/differentiable_training_ablation.py`.

The trained model is legible, recompilable source. After training, w^* is written back into a fresh `.su` as a numeric *literal* with the `w` parameter removed — the trained model expressed as plain Sutra source:

```
function fuzzy rule(vector x, vector own, vector o0, vector o1) {
    return ((1.431431) * similarity(x, own))
        && !((1.431431) * similarity(x, o0))
        && !((1.431431) * similarity(x, o1));
}
```

This baked `.su` is recompiled through the same PyTorch codegen and, fed the same trained prototypes, reproduces logits identical to the parametric-w model (max per-logit difference $\approx 2\times10^{-7}$, attributable to float reassociation) — hence identical 100% accuracy. Round-trip verified for every seed. The trained artifact is therefore not an opaque checkpoint blob: it is a Sutra program whose learned parameter is a literal in the source, and recompiling that source reproduces the trained behaviour. Gradient descent here tunes both the embeddings *and* a

weight that survives as readable code — a trained model that is also legible logic. (Two-seed, single-scale; a verification of the source–training round-trip, not a benchmark.) Reproduction: `experiments/differentiable_training_weighted.py` (trains w + prototypes through the emitted graph, bakes w^* into `.su`, recompiles, and asserts the round-trip).

4.8 Type system and surface syntax — method

Sutra's surface syntax is typed: every value carries a primitive class from a fixed set (`int`, `float`, `complex`, `char`, `bool`, `fuzzy`, `trit`, `vector`, `matrix`, `permutation`, `map`, `string`, `number`, `void`), and the type drives the synthetic-axis allocation in the extended layout (Appendix B). Type information is pre-compile-time annotation in the TypeScript sense: it is read by the inliner and the layout pass before the tensor graph is built, but it is opinionated rather than authoritarian. A divergent assignment warns and still emits a graph, because the runtime guarantee is mathematical not structural; a type mismatch produces a semantically meaningless but mathematically valid output rather than a runtime exception. The surface itself presents `if` / `while` / `for` / assignment forms that read imperatively for ergonomic familiarity; the inliner and egglog simplifier (§4) beta-reduce these into the functional tensor-op core, so the imperative-looking source is a veneer over the same compiled graph a hand-written functional spec would produce.

A concrete source example grounds the surface. The following is the encode/decode core of `examples/role_filler_record.su` verbatim — a role-filler record encoded as one vector and a field decoded back, with no control flow in the program text:

```
function vector make_record(vector name, vector color, vector shape) {
    return bundle(
        bind(r_name,  name),
        bind(r_color, color),
        bind(r_shape, shape)
    );
}

function string decode_field(vector record, vector role) {
    vector recovered = unbind(role, record);
    vector winner = argmax_cosine(
        recovered,
        [f_alice, f_bob, f_red, f_blue, f_circle, f_square]
    );
    return FILLER_NAME[winner];
}
```

`make_record` beta-reduces to one fused tensor expression (each `bind` a constant-matrix matmul, `bundle` a normalized sum); `decode_field` lowers to a single `unbind` matmul plus an argmax-cosine against the codebook. No `bind`/`bundle`/`unbind` symbol or host control flow survives in the compiled graph; Appendix F traces an analogous reduction end-to-end.

---

5 The Sutra Compiler

The compiler is a five-stage pipeline:

1. **Lex + parse**: `.su` source → AST.
2. **Inline + simplify**: stdlib operator definitions inlined; an egglog-based simplifier folds equivalent expressions and runs common-subexpression elimination over the algebra.
3. **Codegen**: AST → Python source emitting PyTorch tensor ops. The emitted module includes the runtime class (`_TorchVSA`) as inline source so the artifact is self-contained.
4. **Compile-time substrate population**: embed_batch fetches embeddings for every string literal; `populate_sutradb` pushes the codebook into SutraDB; `prewarm_rotation_cache` precomputes role rotations.

5. **Execute**: emitted module loaded; chosen device (CUDA or CPU) initialized at module import; `main()` called; result returned.

The runtime class is emitted inline rather than imported because the emitted module *is* the substrate-pure tensor-op graph; every compile-time decision (extended-state-vector dimensions, codebook contents, role rotations, SutraDB path, optional `torch.compile`) is baked into the emitted source. Stages 1–4 run at compile time and stage 5 is the runtime forward pass; Figure~3 diagrams the pipeline as a vertical flow with the residual at each stage.

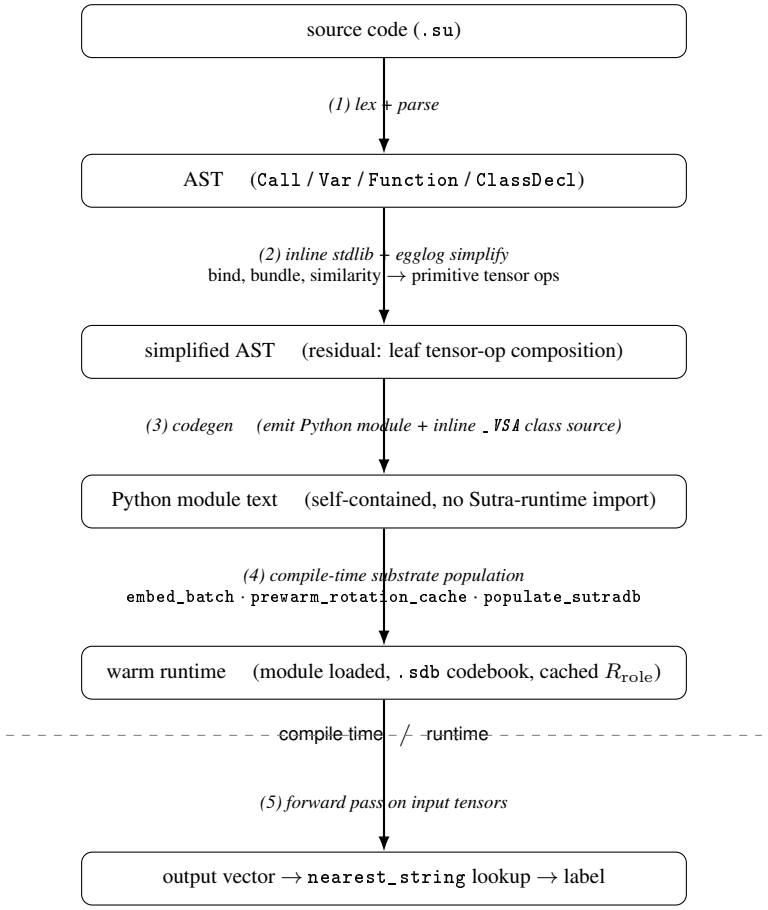

Figure 3: Five-stage compilation pipeline (§4). Boxes are intermediate artifacts; italic labels are the compiler passes that connect them.

5.1 Source-language frontends

The `.su` surface syntax that enters stage 1 is itself a compilation target. Nine source-language frontends — OCaml, TypeScript, Rust, Scala, Clojure, Elixir, Erlang, F#, and Haskell (a C frontend exists but is parked) — each lower their source to `.su` text, which then runs through the same five-stage pipeline unchanged. Each frontend is a pure tree-sitter-based source-to-source pass that never touches the substrate directly; substrate purity is therefore inherited from the single compiler rather than re-established per frontend. If a frontend emits valid Sutra, the result is a substrate-pure tensor-op graph, because the compiler is the only stage that lowers to tensors.

The frontends agree on a small shared set of lowering shapes: functions map to `function` declarations; conditionals, pattern `match`, and guards lower to the soft-mux defuzz blend (no host control flow); tail-recursive accumulators become declared `while_loop` RNN cells (a self-calling function would not terminate through the fuzzy-`if` blend, so the recursion is reified as a loop); foldable non-tail recursion becomes a continuation-passing accumulator trampoline carried by the same loop

construct; and algebraic data — variants, records, tuples, structs — maps to the tagged and structural axons used for bundled records. A new frontend is mostly the work of recognizing each source language's spelling of these shapes.

These are fixture-tested lowering passes of varying maturity (OCaml is the reference at 61 fixtures; the other functional frontends range from 21 to 31, with a parked C frontend), not production compilers, and each fixture is compiled **and run** on the substrate with its output compared to the source language's own ground-truth result — a lowering that parses but computes the wrong value is a failure, not a pass. Recursion outside the two supported shapes, and source constructs with no clean substrate meaning, are surfaced as unsupported rather than mis-lowered.

5.2 Substrate-purity invariants

Three invariants the compiler enforces: (1) every primitive runs on the substrate (numpy is allowed only at compile time for codebook construction and rotation pre-warm, never on the runtime hot path); (2) no scalar extraction inside an operation: operations may not unpack a Python float from a substrate vector, do scalar arithmetic, and pack the result back; (3) no Python control flow inside an operation: loop halt uses substrate primitives (`heaviside`, `saturate_unit`) instead of Python ternaries.

5.3 Compile-time resolution of role rotations

The central compile-time mechanism that lets the compiler emit a substrate-pure tensor-op graph is **precomputed rotation matrices**: every role rotation is constructed at compile time (`prewarm_rotation_cache`) and stored as a constant tensor. At runtime, `bind(role, filler)` is a single matmul against a precomputed matrix, the compile-time resolution eliminates the QR construction from the runtime graph entirely. Role rotations are runtime constants, like neural-network weights at inference; opt-in `torch.compile` (`SUTRA_TORCH_COMPILE=1`) further folds the per-tick loop body into a single fused kernel. Appendix F traces the lowering of `encode2(r_a, f_a, r_b, f_b) := bundle(bind(r_a, f_a), bind(r_b, f_b))` through every reduction stage; the bottom of the chain contains no `bind`/`bundle`/ `normalize` symbol and no Python control flow, so surface lambda calculus and runtime tensor arithmetic are two notations for the same computation.

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6 Demonstration, limitations, and future work

The smoke test (`examples/_smoke_test.py`) runs 10 demonstration programs end-to-end across 27 `.su` files (Appendix I); loop coverage lives in `examples/do_while_adder.su` and the 23-case `test_loop_function_decl.py` suite, and the §3.6 differentiable- training experiment uses the same primitive set. The embedded codebook covers the compile-time embed $\rightarrow$ runtime decode path; extended features (hashmap routing, persistent codebook via `SUTRA_DB_PATH`) are deferred pending a concrete requirement.

6.1 Limitations

The demonstrated regime is bounded, and we state the bounds explicitly:

- **Composition depth.** Decode is exact for single-cycle records but degrades with chained bind/unbind through bundled distractors: 100% through chain length $L = 2$, falling to chance by $L = 8$ on every substrate (§3.2.1). The demonstrated programs are single-cycle; deep nested records are out of the validated regime.
- **Substrate dependence.** Every number is measured on four specific frozen substrates. Large-bundle-width decode capacity varies substantially by substrate (§3.2 table) and is not guaranteed for an arbitrary embedding model; behaviour under model drift is future work.
- **Codebook scale.** The compile-time codebook is $O(\text{vocabulary})$; very large codebooks would require approximate-nearest-neighbour trade-offs not explored here.

- **No external-system benchmark.** We measure rotation versus Hadamard binding and gradient flow through the compiled graph; we do not benchmark against other neuro-symbolic systems (Scallop, DeepProbLog) on a shared task, and we do not benchmark compiler/runtime wall-clock beyond the indicative timings in Appendix H.
- **Binding is fixed, not learned.** Role bindings are content-hash-seeded Haar rotations; learned or semantic binding operators are not part of this work.
- **Training is in-sample.** The §3.6 experiment verifies gradient flow to every learnable parameter, not generalization; no held-out split is reported.

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7 Conclusion

Sutra compiles a typed pure-functional source language to a PyTorch tensor-op graph over the embedding substrate: one vector layout per value, one tensor op per primitive, one dataflow graph per program, no type dispatch at the leaves. Compiled programs are substrate-pure, and the straight-line graph — connectives, binding, similarity — is differentiable end-to-end (gradients verified through the emitted graph, §3.6). One scoping note: loop halt detection uses a hard step function, so gradients do not flow through loop termination — programs containing runtime loops execute forward-only today, and a differentiable halt surrogate is future work. With the language in hand, asking which embedding operations compose at what capacity on which substrates becomes a program to write.

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8 Reproducibility

Sutra — the language, compiler, runtime, and every `.su` program cited in this paper — is openly available at https://github.com/EmmaLeonhart/Sutra. A self-contained replication package is published at https://sutra.topazcomputing.com/sutra-replication-package.zip: it ships `SKILL.md`, an agent-runnable recipe an autonomous agent can follow to install the toolchain and re-derive every result reported in this paper end-to-end, with no human in the loop. The project site (this paper in HTML and the conceptual documentation) is at https://sutra.topazcomputing.com; the paper PDF is at https://sutra.topazcomputing.com/paper.pdf.

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9 AI-use statement

This work was developed in substantial collaboration with large language models, including for ideation, exploration of the vector-symbolic literature, and drafting. The author independently designed and implemented the compiler and runtime, verified that programs execute as substrate tensor operations, ran and checked all experiments, and is responsible for the correctness of every claim, number, and citation in this paper. No results or references were accepted from a model without verification. No experimental result, table, or numerical value reported in this paper was generated by a language model; every number comes from the author-run experiments and reproduction scripts cited in the text and appendices.

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Appendix

Appendix A. Notation: extended layout and primitive operations

We work in a fixed-dimensional real vector space $\mathbb{R}^d$ where d is the substrate's embedding dimension (768 for nomic-embed-text, 384 for all-minilm, 1024 for mxbai-embed-large, 320 for ESM-2). Every Sutra value carries the extended layout $[\,\text{semantic} \mid \text{synthetic}\,]$, a d-dimensional semantic block holding the substrate embedding, concatenated with a small fixed-width synthetic block reserving canonical axes for primitive types (real, imag, truth, char, loop-done) and slot machinery (§3.3). Where notation does not distinguish, "vector" means "the full extended-layout tensor."

The seven primitive operations are:

$$\begin{aligned}
\text{bind}(r, f) &= R_r\, f, \qquad R_r = \text{QR}(\text{seed} = \text{hash}(r)).Q \\
\text{unbind}(r, v) &= R_r^\top v \\
\text{bundle}(x, y) &= \frac{x + y}{\|x + y\| + \varepsilon} \\
\text{similarity}(x, y) &= \frac{x \cdot y}{\|x\|\,\|y\| + \varepsilon} \\
\text{normalize}(v) &= \frac{v}{\|v\| + \varepsilon}
\end{aligned}$$

plus the Lagrange Kleene gates (scalar $\to$ scalar, exact on the $\{-1, 0, +1\}^2$ grid, §1.1-1) and the soft-halt cell (state, halt $\to$ state′, halt′, §3.4).

The Lagrange gates in closed form:

$$\begin{aligned}
\text{AND}(a, b) &= \tfrac{1}{2}\big(a + b + ab - a^2 - b^2 + a^2 b^2\big) \\
\text{NAND}(a, b) &= \tfrac{1}{2}\big(-a - b - ab + a^2 + b^2 - a^2 b^2\big) \\
\text{OR}(a, b) &= \tfrac{1}{2}\big(a + b - ab + a^2 + b^2 - a^2 b^2\big) \\
\text{NOR}(a, b) &= \tfrac{1}{2}\big(-a - b + ab - a^2 - b^2 + a^2 b^2\big) \\
\text{NOT}(a) &= -a \\
\text{XOR}(a, b) &= -ab \\
\text{XNOR}(a, b) &= ab
\end{aligned}$$

The soft-halt cell update is, in compact form,

$$\begin{aligned}
s_{t+1} &= R\, s_t && \text{(rotation step)} \\
h_t &= \text{Heaviside}(\text{cond}(s_t)) && \text{(per-tick halt signal)} \\
H_t &= \text{sat}_{[0,1]}\Big(\textstyle\sum_{k \le t} h_k\Big) && \text{(cumulative monotone halt)} \\
\hat{s}_{t+1} &= H_t\, s_t + (1 - H_t)\, s_{t+1} && \text{(soft-mux freeze)}
\end{aligned}$$

Every right-hand side is a tensor expression with no Python control flow. The compile-time primitives `RotationFor` and `embed` produce constants R_r and basis vectors at compile time and are not part of the runtime tensor graph.

Appendix B. Extended-state-vector layout: per-axis assignments

§3.3 describes the `[semantic | synthetic]` layout in prose. The diagram and per-axis purpose table below give the concrete allocation referenced in `codegen_pytorch.py`:

```
        +-------------------------+----+----+----+----+----+----------+
value   | semantic block          | R  | I  | T  | C  | L  | slots... |
        +-------------------------+----+----+----+----+----+----------+
        |<-- semantic_dim ------->|<--- synthetic_dim ----------------|>
                                    0    1    2    3    4    5..
                                  REAL IMAG TRUTH CHAR LOOP_DONE
                                                  _FLAG
```

Index	Purpose
`synthetic[0]`	`AXIS_REAL` (real component for int/float/complex)
`synthetic[1]`	`AXIS_IMAG` (imaginary component for complex)
`synthetic[2]`	`AXIS_TRUTH` (fuzzy truth scalar; bool/comparisons)
`synthetic[3]`	`AXIS_CHAR_FLAG` (marks char primitives)
`synthetic[4]`	`AXIS_LOOP_DONE` (substrate-side completion flag)
`synthetic[5]`	`AXIS_PROMISE_FULFILLED` (promise status)
`synthetic[6]`	`AXIS_PROMISE_REJECTED` (promise status)
`synthetic[7]`	`AXIS_AXON_POPULATED` (axon arrival flag)
`synthetic[8..]`	reserved headroom (`SLOT_BASE`)

Variable slots do not occupy synthetic axes: each slot stores a full extended-layout vector in a per-slot store keyed by compile-time slot indices (scalars as number-vectors on `AXIS_REAL`; String/vector state carried whole), so every slot type shares one representation.

Appendix C. Capacity: full per-substrate sweeps

Cross-substrate decode accuracy at full bundle widths k ∈ {2, 4, 8, 16, 24, 32, 48}. The four substrates use 84-entry vocabularies (LLM substrates: 84-word noun set spanning animals, foods, objects, places, abstract nouns; ESM-2: 84-sequence amino-acid set covering canonical signal peptides, cell-penetrating peptides, antimicrobial peptides, classic affinity-tag motifs, and deterministic random k-mers). All embeddings are unit-normalized; nomic-embed-text and ESM-2 are additionally mean-centered.

nomic-embed-text (768-d, mean-centered):

k	rotation accuracy	rotation signal cos	Hadamard accuracy	Hadamard signal cos
2	100.0%	+0.703	95.0%	+0.488
4	100.0%	+0.497	95.0%	+0.400
8	100.0%	+0.354	87.5%	+0.307
16	100.0%	+0.251	84.4%	+0.230
24	100.0%	+0.203	60.8%	+0.189
32	99.1%	+0.176	63.1%	+0.167
48	93.3%	+0.144	48.3%	+0.136

all-minilm (384-d):

k	rotation accuracy	rotation signal cos	Hadamard accuracy	Hadamard signal cos
2	100.0%	+0.711	45.0%	+0.386
4	100.0%	+0.506	10.0%	+0.335
8	100.0%	+0.356	7.5%	+0.315
16	92.5%	+0.252	3.1%	+0.299
24	76.2%	+0.203	2.9%	+0.300
32	66.9%	+0.179	2.5%	+0.297
48	42.3%	+0.144	1.7%	+0.294

k	rotation accuracy	rotation signal cos	Hadamard accuracy	Hadamard signal cos

mxbai-embed-large (1024-d):

k	rotation accuracy	rotation signal cos	Hadamard accuracy	Hadamard signal cos
2	100.0%	+0.708	15.0%	+0.311
4	100.0%	+0.500	2.5%	+0.304
8	100.0%	+0.353	2.5%	+0.295
16	98.8%	+0.251	1.2%	+0.294
24	95.8%	+0.203	0.8%	+0.293
32	85.3%	+0.176	0.9%	+0.292
48	72.1%	+0.146	1.0%	+0.291

ESM-2 small protein language model (320-d, mean-centered):

k	rotation accuracy	rotation signal cos	Hadamard accuracy	Hadamard signal cos
2	100.0%	+0.713	75.0%	+0.470
4	100.0%	+0.501	50.0%	+0.323
8	100.0%	+0.349	28.7%	+0.257
16	90.6%	+0.252	16.2%	+0.185
24	77.1%	+0.205	11.2%	+0.171
32	61.9%	+0.174	6.2%	+0.141
48	44.2%	+0.143	4.2%	+0.117

The signal cosine for Hadamard is comparable to rotation's, but the noise floor is much higher because the elementwise product of correlated real-valued embeddings produces a result that overlaps with many distractors in the codebook rather than near-orthogonally with one.

Appendix D. Crosstalk depth: full per-substrate L-sweep

The §3.2.1 protocol: chain length L $\in$ {1, 2, 4, 8, 16, 32}, 20 trials, bundle width 4 (3 distractors per cycle). Forward-bind through L role rotations bundling 3 distractor (role, filler) pairs at each step; unbind in reverse and decode. Two flavors: *raw* (no cleanup) and *snap* (argmax-cosine cleanup against the codebook after each unbind step).

substrate	L=1 raw	L=2 raw	L=4 raw	L=1 snap	L=2 snap	L=4 snap
nomic-embed-text	100%	100%	20%	100%	10%	0%
all-minilm	100%	100%	5%	100%	0%	0%
mxbai-embed-large	100%	100%	5%	100%	0%	0%

By chain length 8 raw accuracy is at chance (1/84) on all three substrates. Snap is *worse* than raw past chain length 1: a hard codebook commitment converts soft noise into a high-confidence wrong answer that the next unbind cannot recover from. The runtime does not implicitly snap between operations; cleanup is an explicit step the program schedules where it knows the codebook is the right reference. Reproduction script: `experiments/crosstalk_chain.py`; raw JSON in `experiments/crosstalk_chain_results.json`.

Appendix E. Codebook implementation details

The §3.5 codebook is implemented as an embedded vector database (internally SutraDB) shipped as part of the compiler, analogous to SQLite being embedded in an application rather than run as a separate service. The data model is RDF triples with f32-vector literals as the object position, indexed by a built-in HNSW index for nearest-neighbor decode. The on-disk format is a `.sdb` file that travels alongside the compiled Python module; no external service, no separate install, no network dependency. Every embedded string in a Sutra program is inserted with the embedding as the object of a triple typed `<http://sutra.dev/f32vec>`. Strings declared but unused in expressions are still inserted, so they remain decodable. The compiled module's Python data section never carries the embeddings, they live in the `.sdb` file, an artifact of compilation, not a service the runtime contacts.

`nearest_string` runs over an HNSW (Hierarchical Navigable Small World) approximate-nearest-neighbor graph maintained by the triplestore. HNSW (Malkov & Yashunin, TPAMI 2020) has **O(log N) expected and worst-case query time** under standard graph-construction parameters; it has displaced linear scan as the default ANN index in Faiss, Milvus, Weaviate, Qdrant, and most production vector databases. A 100-string codebook and a 100,000-string codebook have comparable decode latency at runtime, modulo HNSW's tunable `M` (graph degree) and `ef_search` (beam width); the cost difference is roughly one extra graph hop per $10\times$ growth in N.

Appendix F. Worked lowering of a two-field bundled record

The body §4.2 sketches the lowering of $\text{encode2}(r_a, f_a, r_b, f_b) := \text{bundle}(\text{bind}(r_a, f_a), \text{bind}(r_b, f_b))$. Here we trace each stage with the explicit residual.

Stage 1: AST after parse. A tree of `Call` nodes over named identifiers: `Call("bundle", Call("bind", r_a, f_a), Call("bind", r_b, f_b))`.

Stage 2: beta reduction by stdlib inlining. `bind`, `bundle`, and `normalize` are stdlib functions: $\text{bind}(r, f) \equiv \text{RotationFor}(r)\, f$, $\text{bundle}(x, y) \equiv \text{normalize}(x + y)$, $\text{normalize}(v) \equiv v/(\|v\| + \varepsilon)$. After substitution the body becomes

$$\text{normalize}\big(\text{RotationFor}(r_a)\, f_a \;+\; \text{RotationFor}(r_b)\, f_b\big).$$

No `bind` or `bundle` symbol remains; the residual is straight- line algebra over four tensor primitives.

Stage 3: compile-time constant resolution. $\text{RotationFor}(r)$ is a compile-time function returning $R = \text{QR}(\text{seed} = \text{hash}(r)).Q$. The compiler evaluates it for each role at compile time, freezes the results as constant tensors R_a and R_b, and stores them in the rotation cache. The body becomes $\text{normalize}(R_a\, f_a + R_b\, f_b)$, R_a and R_b are now load-bearing constants in the same sense as the weight matrices of a feed-forward network.

Stage 4: peephole fusion. The simplifier recognizes $\text{normalize}\big(\sum_i R_i\, f_i\big)$ as the bundle-of-binds pattern and rewrites it to `_VSA.bundle_of_binds([(R_a, f_a), (R_b, f_b)])`, one kernel launch instead of two matmuls + add + norm.

Stage 5: leaf tensor ops at runtime. `bundle_of_binds` stacks rotations into a (k, d, d) tensor, stacks fillers into (k, d), runs one batched einsum + sum + L2-normalize:

$$v \;=\; \sum_k R_k\, f_k \;=\; \texttt{einsum}(\text{"}\texttt{kij,kj->i}\text{"},\ \text{stack}([\texttt{R}_\texttt{a}, \texttt{R}_\texttt{b}]),\ \text{stack}([\texttt{f}_\texttt{a}, \texttt{f}_\texttt{b}]))$$

$$\text{encode2} \;=\; v \,/\, (\|v\| + \varepsilon)$$

The compiled forward pass for `encode2` is exactly those three torch calls (einsum, linalg.norm, divide) over precomputed R_a, R_b and runtime-supplied f_a, f_b.

Appendix G. §3.6 differentiable-training vocabulary

Twenty categories of fifty words each (992 unique after deduplication), embedded via nomic-embed-text:

- **animal**: dog, cat, bird, fish, horse, lion, tiger, elephant, rabbit, monkey, bear, wolf, fox, deer, mouse, snake, frog, turtle, dolphin, whale, shark, eagle, owl, sparrow, crow, robin, parrot, swan, duck, goose, chicken, cow, pig, sheep, goat, donkey, camel, giraffe, kangaroo, koala, panda, leopard, cheetah, hippopotamus, rhinoceros, antelope, buffalo, hedgehog, squirrel, raccoon
- **vehicle**: car, truck, airplane, boat, bicycle, motorcycle, bus, train, ship, helicopter, tractor, scooter, van, taxi, jeep, sailboat, kayak, canoe, raft, submarine, glider, jet, rocket, spaceship, sled, skateboard, wagon, carriage, chariot, ambulance, firetruck, limousine, minivan, hatchback, sedan, coupe, convertible, pickup, trailer, ferry, yacht, dinghy, blimp, balloon, hovercraft, tram, moped, tricycle, rollerblade, unicycle
- **food**: apple, bread, cheese, rice, pasta, banana, salad, soup, meat, pizza, sandwich, burger, taco, sushi, cake, cookie, pie, donut, muffin, pancake, waffle, bagel, croissant, omelet, salmon, tuna, beef, pork, lamb, bacon, ham, sausage, steak, lobster, shrimp, crab, oyster, clam, broccoli, carrot, lettuce, tomato, potato, cucumber, onion, garlic, pepper, eggplant, spinach, mushroom
- **color**: red, blue, green, yellow, orange, purple, black, white, brown, pink, gray, cyan, magenta, violet, indigo, turquoise, teal, lavender, maroon, crimson, scarlet, ruby, gold, silver, bronze, copper, beige, tan, ivory, charcoal, navy, sapphire, emerald, jade, olive, lime, mint, coral, peach, plum, mauve, fuchsia, amber, ochre, sienna, mahogany, chocolate, caramel, mustard, azure
- **clothing**: shirt, pants, dress, hat, shoes, jacket, socks, gloves, scarf, belt, sweater, hoodie, jeans, shorts, skirt, blouse, coat, cap, beanie, mittens, tights, leggings, vest, blazer, suit, tuxedo, gown, robe, kimono, kilt, poncho, cloak, cape, sneakers, boots, sandals, slippers, heels, loafers, tie, bowtie, cufflinks, watch, ring, necklace, earrings, bracelet, anklet, brooch, headband
- **weather**: rain, snow, wind, cloud, storm, fog, frost, hail, thunder, lightning, drizzle, downpour, blizzard, hurricane, tornado, cyclone, typhoon, sleet, mist, haze, smog, sunshine, sunlight, sunset, sunrise, dawn, dusk, twilight, breeze, gust, gale, humidity, drought, flood, monsoon, snowfall, snowstorm, rainstorm, sandstorm, heatwave, chill, dew, hailstorm, thaw, overcast, sunny, cloudy, rainy, snowy, windy
- **emotion**: joy, sadness, anger, fear, love, hope, surprise, disgust, pride, envy, happiness, grief, rage, anxiety, affection, despair, delight, shame, guilt, confidence, contentment, jealousy, regret, sorrow, frustration, satisfaction, awe, wonder, gratitude, compassion, sympathy, empathy, irritation, boredom, excitement, enthusiasm, calm, serenity, melancholy, nostalgia, longing, embarrassment, humiliation, indifference, ecstasy, bliss, dread, terror, amusement, loneliness
- **tool**: hammer, saw, drill, wrench, screwdriver, knife, scissors, pliers, axe, shovel, rake, hoe, spade, pickaxe, crowbar, mallet, chisel, sander, level, ruler, vise, clamp, ratchet, socket, awl, scraper, trowel, broom, mop, sponge, bucket, ladder, jackhammer, sledgehammer, paintbrush, roller, stapler, tongs, tweezers, calipers, magnifier, flashlight, multimeter, wirecutter, hacksaw, router, torch, soldering_iron, drillbit, screwbit
- **instrument**: guitar, piano, drum, violin, flute, trumpet, saxophone, harp, cello, clarinet, banjo, mandolin, ukulele, harmonica, accordion, organ, keyboard, synthesizer, xylophone, tambourine, maracas, bongos, marimba, vibraphone, glockenspiel, bagpipes, oboe, bassoon, trombone, tuba, lute, sitar, koto, zither, dulcimer, cymbal, gong, triangle, cowbell, snare, kettledrum, recorder, piccolo, fife, didgeridoo, theremin, viola, double_bass, fiddle, ocarina
- **profession**: doctor, teacher, lawyer, engineer, nurse, chef, artist, scientist, farmer, plumber, electrician, carpenter, mechanic, pilot, sailor, soldier, judge, journalist, writer, poet, painter, sculptor, musician, actor, dancer, singer, photographer, architect, dentist, surgeon, pharmacist, veterinarian, librarian, accountant, banker, broker, programmer, designer, manager, secretary, butcher, baker, gardener, tailor, jeweler, barber, chemist, biologist, physicist, mathematician
- **body_part**: head, hand, foot, eye, ear, nose, mouth, leg, arm, finger, toe, knee, elbow, shoulder, hip, neck, back, chest, stomach, heart, brain, lung, liver, kidney, bone, muscle, skin, hair, throat, jaw, chin, cheek, forehead, eyebrow, eyelash, lip, tongue, palm, wrist, ankle, thumb, heel, spine, rib, scalp, nostril, gum, knuckle, tendon, vein
- **plant**: tree, flower, grass, bush, vine, fern, moss, herb, weed, leaf, stem, branch, bark, blossom, petal, oak, maple, willow, birch, cedar, bamboo, cactus, rose, tulip, daisy, lily,

sunflower, orchid, ivy, basil, rosemary, thyme, sage, lavender, dandelion, clover, lotus, magnolia, sycamore, redwood, baobab, eucalyptus, juniper, hemlock, fir, spruce, ash, elm, poplar, chestnut

- **furniture**: chair, table, sofa, bed, desk, shelf, drawer, cabinet, wardrobe, dresser, nightstand, ottoman, bench, stool, recliner, futon, couch, armchair, bookcase, sideboard, buffet, cupboard, hutch, vanity, headboard, footboard, mattress, pillow, cushion, blanket, quilt, comforter, lamp, mirror, rug, carpet, curtain, blind, shutter, hammock, cradle, crib, bassinet, highchair, rocker, loveseat, settee, divan, chaise, headrest
- **building**: house, apartment, mansion, cottage, cabin, hut, igloo, tent, palace, castle, fortress, tower, skyscraper, office, factory, warehouse, store, mall, restaurant, hotel, motel, hospital, school, university, library, museum, theater, stadium, arena, church, temple, mosque, synagogue, cathedral, chapel, monastery, abbey, barn, shed, garage, basement, attic, cellar, lobby, lounge, hallway, corridor, atrium, foyer, balcony
- **country**: France, Germany, Italy, Spain, Portugal, England, Scotland, Ireland, Norway, Sweden, Finland, Denmark, Iceland, Russia, Poland, Greece, Turkey, Egypt, Morocco, Algeria, Kenya, Nigeria, Ethiopia, Ghana, Senegal, Mali, Sudan, Uganda, Tanzania, Madagascar, China, Japan, Korea, Vietnam, Thailand, Malaysia, Indonesia, India, Pakistan, Bangladesh, Iran, Iraq, Israel, Lebanon, Australia, Canada, Mexico, Brazil, Argentina, Chile
- **sport**: football, basketball, baseball, soccer, tennis, golf, hockey, rugby, cricket, volleyball, swimming, running, cycling, skiing, snowboarding, surfing, sailing, rowing, kayaking, climbing, hiking, boxing, wrestling, fencing, archery, shooting, fishing, hunting, polo, badminton, ping_pong, squash, racquetball, lacrosse, handball, dodgeball, kickball, gymnastics, diving, weightlifting, judo, karate, taekwondo, sumo, marathon, triathlon, decathlon, biathlon, skating, bowling
- **drink**: water, juice, milk, tea, coffee, soda, beer, wine, whiskey, vodka, rum, gin, tequila, brandy, cognac, champagne, cocktail, smoothie, milkshake, lemonade, cider, ale, lager, stout, bourbon, scotch, sake, mead, punch, eggnog, kombucha, kefir, espresso, latte, cappuccino, mocha, americano, macchiato, frappe, hot_chocolate, cordial, shake, slushie, syrup, fizz, brew, tonic, infusion, ginger_ale, root_beer
- **metal**: gold, silver, copper, iron, steel, aluminum, brass, bronze, tin, lead, zinc, nickel, platinum, titanium, chromium, mercury, magnesium, lithium, sodium, potassium, calcium, uranium, plutonium, palladium, tungsten, vanadium, cobalt, manganese, beryllium, gallium, indium, antimony, bismuth, cadmium, cerium, neodymium, osmium, rhodium, ruthenium, tantalum, thallium, thorium, yttrium, scandium, hafnium, niobium, molybdenum, rhenium, iridium, rubidium
- **shape**: circle, square, triangle, rectangle, oval, ellipse, pentagon, hexagon, octagon, diamond, rhombus, trapezoid, parallelogram, polygon, sphere, cube, cylinder, cone, pyramid, prism, cuboid, tetrahedron, dodecahedron, icosahedron, octahedron, torus, helix, spiral, crescent, star, heart, arrow, cross, line, curve, arc, ring, loop, knot, dot, vertex, edge, angle, parabola, hyperbola, sine, wave, zigzag, scallop, annulus
- **fabric**: cotton, wool, silk, linen, polyester, nylon, denim, leather, suede, velvet, satin, lace, tweed, cashmere, mohair, fleece, fur, canvas, burlap, jute, flannel, chiffon, organza, taffeta, brocade, damask, paisley, gingham, plaid, herringbone, corduroy, microfiber, spandex, lycra, rayon, viscose, acrylic, polypropylene, jersey, knit, sherpa, gabardine, twill, muslin, gauze, mesh, vinyl, tulle, georgette, voile

Appendix H. Reproduction details and hyperparameters

Per-experiment configuration. All scripts live under `experiments/` in the source repository; each writes a JSON results file to the same directory on completion. RNG seeds are fixed in the source files cited; re-running reproduces the numbers reported in the body to the precision reported.

- **Rotation vs Hadamard, LLM** (§3.2) — script `rotation_binding_capacity_llm.py`; 10 trials per k; substrates nomic-embed-text, all-minilm, mxbai-embed-large; seeds fixed per script.
- **Rotation vs Hadamard, ESM-2** (§3.2) — script `rotation_binding_capacity_bioinformatics.py`; 10 trials per k; substrate `facebook/esm2_t6_8M_UR50D`; seeds 1729 and 2718.

- **Crosstalk depth** (§3.2.1) — script `crosstalk_chain.py`; 20 trials per chain length L; three LLM substrates; seeds fixed per script.
- **Differentiable training** (§3.6) — script `differentiable_training_compiled.py --batched`; 3 seeds × 30 epochs (K=5, 50 words); substrate nomic-embed-text (frozen); optimizer Adam, lr=0.01; seeds 0–2.
- **Trained weight → legible source** (§3.7) — script `differentiable_training_weighted.py`; 2 seeds × 30 epochs (K=3, 24 words); substrate nomic-embed-text (frozen); optimizer Adam, lr=0.02; seeds 0–1.

The differentiable-training run (`differentiable_training_compiled.py`) generates a `.su` fuzzy-rule classifier, compiles it with the PyTorch codegen, and backpropagates five randomly-initialized unit-normalized prototype vectors through the *emitted* `rule` function (the compiler's `_VSA.similarity` composed with the Lagrange–Kleene polynomials). Five classes, ten words each (50 inputs); embeddings are precomputed once and cached to `.diff_train_embeddings.pt` so reruns skip the embed step. A build-time assertion checks the emitted `similarity` is not `float()`-collapsed (Stage A0), so gradients provably flow through the compiled graph rather than a reimplementation. The forward is evaluated batched via `torch.vmap` over that same emitted `rule`; a runtime assertion requires the batched and per-sample evaluations to agree within 10^{-4} before training, so the fast path is the identical compiled computation, not a substitute.

The §3.7 run (`differentiable_training_weighted.py`) adds a `number w` parameter to the `.su` rule, trains w together with the prototypes through the same emitted graph, then regenerates the rule with w^* substituted as a numeric literal and the parameter removed, recompiles that source through the PyTorch codegen, and asserts the recompiled program's logits match the parametric model (max per-logit difference < 10^{-4}) — a source-level training round-trip, not a benchmark.

Hardware used for the numbers in the body: CPU torch on a single laptop (no CUDA). The §3.6 compiled run takes ≈ 230 s (K=5, 50 words, 30 epochs, 3 seeds) via the batched `torch.vmap` path over the same emitted ops; the equivalent per-sample Python driver produces the bit-identical result but takes ≈ 6.2 h — an interpreter-overhead artifact, not a cost of the compiled graph; the §3.7 weighted round-trip (K=3, 24 words, 30 epochs, 2 seeds, per-sample — small enough that the driver cost is ≈ 2.5 min) completes including the recompile check; the §3.2 capacity sweeps complete in ~2 min per substrate; the §3.2.1 crosstalk sweep completes in ~5 min. Re-running on CUDA should reproduce the same accuracy numbers since the operations are deterministic given a seed.

Appendix I. Demonstration corpus

The smoke test (`examples/_smoke_test.py`) compiles and runs ten `.su` programs end-to-end and asserts each output against a hardcoded expected value. The programs collectively exercise the language features the body claims, with no Python control flow on the runtime path:

Program	Feature exercised
`hello_world.su`	embed + retrieve (minimal program)
`fuzzy_branching.su`	weighted-superposition conditional
`role_filler_record.su`	bind / bundle / unbind on a 3-field record (§2.1)
`classifier.su`	cosine-similarity classifier over a small codebook
`analogy.su`	associative pair memory: capital → country recovery via unbind
`knowledge_graph.su`	(subject, relation, object) triple encode + decode
`predicate_lookup.su`	bind-keyed dictionary read
`fuzzy_dispatch.su`	Lagrange-Kleene-gated dispatch among handlers
`nearest_phrase.su`	top-1 phrase retrieval over a `.sdb` codebook
`sequence.su`	foreach reduction over a list

Loop coverage lives in `examples/do_while_adder.su` and the 23-case `tests/test_loop_function_decl.py` suite. The §3.6 differentiable-training experiment

uses the same primitive set the smoke-test programs are built from, no Sutra-runtime extensions, just compilation of `.su` source to PyTorch tensor ops.